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The zero set of the order parameter in two dimensions

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Abstract
We survey the possible zero sets of the order parameter wavefunction in two-dimensional domains. Certain structures are feasible, including the well-known Abrikosov lattice, the giant vortex and the line cut (in multiply connected domains). Other proposed structures such as the vortex ring and the noninteger vortex point are shown to be impossible.

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1. Introduction

The zero set of the superconducting order parameter is one of the main pattern of interest to theoreticians and experimentalists alike. The story of the zero set starts with Abrikisov's [1] theoretical prediction of vortices. Some other forms of zero sets were predicted in recent years. The purpose of this paper is to point out the possible forms that the zero set can take in two-dimensional samples. As we shall see, some of the recently conjectured zero sets are mathematically impossible. On the other hand, other forms of the zero set are mathematically rigorous, but were not observed yet experimentally.

We consider a two-dimensional superconducting sample occupying a region D in R^2. We use the following nondimensional form of the Ginzburg Landau (GL) model for the energy functional:

$$G(u, \mathbf{A}) = \int_D \left(|(\mathrm{i}\nabla - \mathbf{A})u|^2 + \frac{1}{2}(|u|^2 - 1)^2 \right) \mathrm{d}x + \kappa^2 \int_{R^2} |\nabla \times \mathbf{A} - \mathbf{H_e}|^2 \,\mathrm{d}x. \tag{1.1}$$

Here D is the domain occupied by the superconducting sample, u is the order parameter wavefunction, H_e is the applied magnetic field, A is the magnetic vector potential and κ is the GL parameter. The critical points of the functional are the solutions of the GL differential equations.

A first rigorous classification of the zero set was performed by Elliott *et al* [11]. They showed that the zero set of global minimizers consists only of isolated points and curves. They further showed that in simply connected domains the zero set of a global minimizer contains only isolated points. Some of the results below are extensions of this classification for the case of local minimizers.

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2. Abrikosov vortices

The first work on the zero set in two-dimensional domains was Abrikosov's seminal paper [1]. He considered a linearized GL model over the entire plane, and showed that for κ larger than the critical GL parameter there exist solutions u that vanish at points located on a rectangular lattice. This is the celebrated Abrikosov vortex lattice. He also found that either all the vortices have circulation (degree) +1, or all of them have circulation -1. His analysis did not exclude other possible lattices. Later, Chapman [9] and Almog [2] settled the case of other periodic arrangements of vortices.

Abrikosov's solution only applies to the case where $D = R^2$. Several rigorous approaches were advanced to study the fully nonlinear model in bounded domains. Thus, Sandier and Serfati [24, 25] (and others) proved the existence of solutions to the GL model in arbitrary domains that posses a large number of vortices, again all of which of circulation +1 or -1. From a different point of view, Almog [3], Pan [22] (and others) considered the loss of stability of the deGennes' surface superconductivity solution as the applied magnetic field is decreased from the critical value H_{c3}. The breakdown of this solution is accompanied by the formation of large arrays of vortices, that are similar in the bulk of D to the Abrikosov lattice. In addition to the vast amount of theoretical investigations related to the classical Abrikosov vortices, they were observed in many experimental setups. From the perspective of this paper these vortices are the 'classical' zero sets of the GL model.

Are there other kinds of zero sets? Four such sets were proposed in the last decade. We shall now consider each one of them.

3. The giant vortex

The existence of a vortex carrying many flux quanta was first predicted, to the best of my knowledge, by Fink and Presson [12]. For many years this possibility was neglected, until it was rediscovered about 10 years ago independently by Bauman *et al* [4] and by Moshchalkov *et al* [20]. Both groups considered mesoscopic domains. For example, Bauman *et al* showed that when D is a mesoscopic disc, and when the applied magnetic field is sufficiently large, there exists a stable solution to the GL equations that has a zero at the disc centre, whose circulation is larger than 1 (in absolute value). Moreover, they provided asymptotic expressions for the circulation as the applied field's strength tends to infinity. Such solutions are called 'giant vortices'.

An extensive theory for giant vortices was provided by Gustafson and Sigal [15]. Furthermore, giant vortices have been verified experimentally by several groups [13, 20, 23] in a variety of setups. Giant vortices can be considered now as established as the Abrikosov lattice, and therefore we proceed to more exotic zero sets.

4. The line cut

The extensive theoretical and experimental activity on mesoscopic multiconnected domains was greatly influenced by the Little Parks effect [18]. Consider, for example, the case of a narrow ring. It seems at first sight that the absolute value of the order parameter would be very close to unity everywhere in the ring. Berger and Rubinstein [6] have shown, however, that even the slightest deviation from uniformity in the ring thickness implies that for discrete flux values the order parameter would vanish at a selected point in the limit of very thin rings.

Later Berger and Rubinstein [8] considered the case of fully two-dimensional domains. They showed that when the magnetic flux through the hole bounded by the ring is (in the appropriate nondimensional units) the fundamental flux quantum plus $1/2$, and when the applied magnetic field is nonzero only inside the hole bounded by the ring, the order parameter vanishes along a line cutting from the inner boundary of the ring to the outer boundary of the ring. An alternative proof of this result was provided by Hellfer *et al* [16]. The geometric tools of [16] made it possible to predict similar zero sets consisting of line cuts in many arrangements of samples with holes.

As far as I know this kind of zero set has not yet been directly confirmed in the lab. There is, however, ample indirect evidence for it [5]. For example, [7] predicted that in nonuniform thin rings the ac magnetic susceptibility would diverge at critical values of the temperature and the magnetic flux. These critical values are exactly those where the line cut is predicted to appear, and the line cut is directly related to the divergence. Indeed such a behaviour was observed in the Zhang–Price experiment [29]. Another prediction of [6] concerns the Little Parks oscillations in the $H(T)$ phase transition curve. The oscillations are the result of the system transition between states with different circulations. The classical data show cusps associated with the oscillations, indicating a discontinuous transition between such states. Since [6] predicts that for some parameters the transition is smooth, we should expect a 'rounding' of the cusps. Indeed this prediction was verified experimentally by Morelle *et al* [21].

5. The vortex ring

A number of researchers reported in recent years on another form of a co-dimension one zero set. The *ring vortex* (sometimes called the 'annular vortex') seems to have been first proposed by Govaerts *et al* [26, 27]. Another type of a ring vortex was constructed by Zhao *et al* [30]. Both groups considered radial symmetric solutions in discs or symmetric rings. The zero set is a circle with the same centre as the disc or the annulus. Thus the zero set divides the disc (or the ring) into two subdomains. In the solution of [26, 27] the circulation is the same in the two subdomains. The solution of Zhao *et al*, on the other hand, has different circulations in the inner and outer subdomains. Thus, all these groups constructed solutions of the form

$$u(r,\theta) = \begin{cases} \rho(r)\,\mathrm{e}^{\mathrm{i}\phi_1(\theta)} & r < R_c \\ \rho(r)\,\mathrm{e}^{\mathrm{i}\phi_2(\theta)} & r > R_c \end{cases} \tag{5.1}$$

where $\rho(R_c) = 0$.

I assume here and in section 6 that the applied magnetic field is a real analytic function, namely, a function that is infinitely differentiable and has a convergent Taylor expansion at every point. This includes of course the common case of constant applied fields. Actually, the results in this section hold also under much weaker smoothness assumptions. I show that the vortex ring functions mentioned above are not stable or metastable. My arguments do not rely on the radial symmetry. In fact, I prove the following general result:

Proposition 5.1. *Let (u, A) be a local minimizer of the GL energy. Then the zero set of u cannot partition D into two parts.*

Proof. The key point in the analysis is that a critical point of the GL energy functional must be a real analytic function [19]. Elliott *et al* [11] used this fact to prove that if the zero set contains a line, then this line must be smooth. They further showed that the zero set of the *global* minimizer cannot partition D into two parts. I shall now extend the result to local minimizers as well. Before doing so, we should recall that a wavefunction u and a vector potential A form a local minimizer of G if $G(u, A) \leqslant G(w, B)$ for all pairs (w, B) that are 'near' (u, A) in a suitable norm. The appropriate norm [10] for the wavefunction u is the H_1 norm, defined as

$$||u||_{H_1} = \int_D |\nabla u|^2 + |u^2|.$$

Assume in contrast that the zero set of a local minimizer u contains a line Γ that divides D into two parts D_1 and D_2, such that $D_1 \bigcup D_2 = D$. We define a new function v such that

$$v(x, y) = \begin{cases} u(x, y) & (x, y) \in D_1 \\ \mathrm{e}^{\mathrm{i}\delta} u(x, y) & (x, y) \in D_2 \end{cases} \tag{5.2}$$

where δ is an arbitrary constant. Clearly v is continuous in D. Moreover $G(u, A) = G(v, A)$. The normal derivative of u across Γ cannot vanish everywhere (otherwise u would vanish identically in D). Because of the phase factor $\mathrm{e}^{\mathrm{i}\delta}$, however, the normal derivative of v at Γ is not continuous. Therefore, v cannot be a local minimizer, and there must be a function in a small H_1 neighbourhood of v that has a lower energy. But since $||u - v||_{H_1}$ can be made as small as we wish by choosing sufficiently small δ, it follows that u cannot be a local minimizer either. □

While the proposition above implies that the annular vortices are not physically attainable, the vortex ring of [30] cannot be a solution of the GL equation at all. This statement follows from a general argument, saying that the phase must change by an odd multiple of π when we cross a line Γ in the zero set.

To see this principle in the special case of the symmetric (radial) solution of [30], consider a wavefunction u of the form (5.1), with $\phi_1 = n\theta + c_1$, and $\phi_2 = m\theta + c_2$, where n, m are the two (different!) circulations on the two sides of the vortex Γ, and c_1 and c_2 are constants. Now, at Γ we must have continuity of $\frac{\partial u}{\partial r}$, implying

$$\left.\frac{\mathrm{d}\rho}{\mathrm{d}r}\,\mathrm{e}^{\mathrm{i}n\theta + c_1}\right|_{\Gamma^-} = \left.\frac{\mathrm{d}\rho}{\mathrm{d}r}\,\mathrm{e}^{\mathrm{i}m\theta + c_2}\right|_{\Gamma^+} \tag{5.3}$$

where we used $\rho(R_c) = 0$, and we denote the two sides of Γ by $\pm$. Continuity of $\frac{\partial u}{\partial r}$ at Γ implies $\left|\frac{\partial u}{\partial r}\right|_{\Gamma^-} = \left|\frac{\partial u}{\partial r}\right|_{\Gamma^+}$, This implies in turn

$$\left.\frac{\mathrm{d}\rho}{\mathrm{d}r}\right|_{-} = -\left.\frac{\mathrm{d}\rho}{\mathrm{d}r}\right|_{+} \tag{5.4}$$

since ρ is decreasing in $r < R_c$ and increasing in $r > R_c$ near Γ. Thanks to these arguments we see that the phase must change by an odd multiple of π as we cross the zero circle Γ; but this contradicts the solution of [30] where the phase jump across Γ changes continuously with θ.

6. Noninteger point vortices

In contrast to the vortex carrying several flux quanta, and thus associated with a degree greater than one, Govaerts *et al* proposed recently ([14, 28]) a vortex with a fractional degree. He used a numerical variational (optimization) method to construct a stable solution that has a singularity of degree 1/2 at the centre of a disc of some radius R. The zero set in this solution consisted, in addition to the origin itself, also of the ray along the x axis connecting $(0, 0)$ with the point $(R, 0)$. I shall now show that this solution is not correct, and in fact, no such half integer vortices can be solutions of the GL equations. In particular they cannot be local minimizers.

In fact, Elliott *et al* [11] proved that the zero set of the *global* minimizer wavefunction $u(x, y)$ cannot contain a line that starts in the interior of the domain. It is not difficult to show that their proof also holds for any critical point of the GL energy. To render the paper self-contained, and since the general argument of [11] relies on deep results from the theory of algebraic curves, I shall present instead a very simple argument showing the infeasibility of the specific construction of [14]. Again I use the fact that a critical point $u(x, y)$ must be a real analytic function. Let us write u as a complex valued function $u(x, y) = \alpha(x, y) + \mathrm{i}\beta(x, y)$. Since α and β both vanish identically for $y = 0, 0 \leqslant x \leqslant R$, we have

$$\alpha(0,0) = \beta(0,0) = \frac{\partial^j}{\partial x^j}\alpha(0,0) = \frac{\partial^j}{\partial x^j}\beta(0,0) = 0 \qquad j = 1, 2, \ldots .$$

But then the analyticity of α and β implies that $\alpha(x, 0)$ and $\beta(x, 0)$ must vanish also for negative values of x, and therefore the zero set must contain also the ray along the negative x axis.

7. Summary

We surveyed the possible structure of the zero set of the superconducting wavefunction. Elliot *et al* [11] proved that the zero set of any solution of the GL equations in two dimensions must consist of isolated points and curves (they wrote specifically on global minimizers, but many of their results carry over to any solutions of the GL equations). The case of isolated points is well known, either in the form of Abrikosov's vortices, or in the form of the giant vortex. Under certain special configurations, there exist stable states where the zero set consists of a line cut, a line connecting two components of the exterior of the sample geometry. We have shown that some recently proposed zero sets, such as a set containing a vortex with noninteger circulation, or a set containing a line that divides the domain into two parts, cannot be stable or metastable. We point out some further interesting constraints on the solution of the GL equations: if the domain is convex, and there is no applied field, there do not exist nonconstant metastable states [17].

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